The Dirac monopole and differential characters.

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Abstract

We describe the Dirac monopole using the Cheeger-Simons differential characters. We comment on the rôle of the Dirac string and on the connection with Deligne cohomology.

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The apparent lack of symmetry of the Maxwell equations $\nabla \cdot \mathbf{E} = \rho_E$ and $\nabla \cdot \mathbf{B} = 0$ under the duality transformation $\mathbf{E} \to \mathbf{B}$, $\mathbf{B} \to -\mathbf{E}$ inspired Dirac [1], [2] to study the consequences of the existence of a hypothetical magnetic charge density ρ_M. For $\rho_M = 4\pi g \delta(r)$, by analogy to Coulomb's law, the magnetic field produced would be $\mathbf{B} = g r^{-3} \mathbf{r}$. The magnetic flux on a 2-sphere S^2 surrounding that magnetic charge would then be

$$\Phi = \int_{S^2} \mathbf{B} \cdot \mathbf{dS} = 4\pi g \tag{1}$$

Classical electrodynamics can also be expressed in terms of the vector potential $\mathbf{A}$ defined by $\mathbf{B} = \nabla \times \mathbf{A}$. Then

$$\Phi = \int_{S^2} \mathbf{B} \cdot \mathbf{dS} = \int_{\partial S^2} \mathbf{A} \cdot \mathbf{dl} = 0 \tag{2}$$

if $\mathbf{A}$ is assumed smooth everywhere on S^2. The apparent contradiction of (1) and (2) compels somebody to accept that either there is no globally defined choice for $\mathbf{A}$ or that the vector potential $\mathbf{A}$ must have a line singularity which starts from the location of the monopole and stretches to infinity. Most of the treatments of this system take the former view since the latter introduces a line singularity ("Dirac string") which is non-physical, as can be demonstrated by the fact that a gauge transformation changes its location. We present an alternative description of the Dirac string by using the Cheeger-Simons differential characters. These objects incorporate singularities so they generalize singular cohomology, thus providing more flexible algebraic objects useful in the description of topological defects.

The conventional approach is to model the Dirac string as a principal $U(1)$ bundle over S^2. The magnetic monopole charge g distinguishes the different ways this vector

bundle can be twisted, so it represents an element of $H^2(S^2,\mathbb{Z})$. The generator of this group, when the corresponding 1st Chern number of the bundle is ±1, is the first integral Chern class, which provides a topological obstruction to the bundle being trivial. The vector potential $\mathbf{A}$ induces an isomorphism $H^2(S^2,\mathbb{Z})\rightarrow H^2(S^2,\mathbb{R})$. The generator of the latter group, when the corresponding 1st Chern number of the bundle is equal to ±1, is the first real Chern class. This provides an obstruction to the corresponding potential being globally flat, namely $dA=0$ for the $u(1)$-valued 1-form A corresponding to $\mathbf{A}$. Since the two cohomology groups are isomorphic, due to the torsion-free nature of $H^2(S^2,\mathbb{Z})$ the two obstructions are equivalent and are used interchangeably [3].

Instead of taking the conventional approach, we may insist on the existence of a globally defined vector potential $\mathbf{A}$. Then $\mathbf{A}$, as well as the corresponding connection A, will have to have line singularities. The theory of singular connections has been studied extensively in [4]. Consider a singular $u(1)$-valued one-form A which is defined on the complement of a point in $\mathbb{R}^3$ i.e. on $\mathbb{R}^3-\{0\}$, the monopole location being excluded from $\mathbb{R}^3$ and let $F=dA$ be the corresponding curvature. Consider also two cycles C and C' on $\mathbb{R}^3-\{0\}$ that do not pass through the monopole. Let S be a 2-chain such that $C'=C+\partial S$. Here C and C' are simplicial models of the circle, S is a simplicial model of a 2-dimensional surface and ∂ denotes the simplicial boundary operator. The simplicial model of the surface of interest should be a 2-chain, and not a 2-cycle, otherwise $\partial S=\emptyset$. Then

$$\int_{C'} A=\int_{C} A+\int_{\partial S} F \tag{3}$$

When an external electromagnetic field is minimally coupled to matter, the wave-

function describing the matter obtains an extra phase $\exp(i\int_S A)$. The quantity of physical significance therefore, is the $\bmod\ \mathbb{Z}$ reduction of (3). We are interested, therefore, in singular connections that obey the relation

$$\int_{C'} A = \int_C A + \int_{\partial S} F \mod \mathbb{Z} \tag{4}$$

This, by definition, means that A is an element of the first differential character group $\hat{H}{}^1(\mathbb{R}^3 - \{0\}, \mathbb{R}/\mathbb{Z})$. A differential character then, is an object that can be defined (non-canonically) by a differential form with singularities [5]. The non-canonical realization encodes the non-uniqueness, and the subsequent lack of physical meaning, of the Dirac string. This construction can be straightforwardly generalized to obtain k-dimensional differential characters. These are elements of the k-th differential character group $\hat{H}{}^k(\mathbb{R}^3 - \{0\}, \mathbb{R}/\mathbb{Z})$ [6]. Since S^2 is a deformation retract of $\mathbb{R}^3 - \{0\}$ [7], we can use S^2 instead of $\mathbb{R}^3 - \{0\}$ in the calculation of the cohomology groups. It turns out that there is an exact sequence [6]

$$0 \longrightarrow H^k(S^2, \mathbb{R}/\mathbb{Z}) \longrightarrow \hat{H}{}^k(S^2, \mathbb{R}/\mathbb{Z}) \longrightarrow \Lambda_o^{k+1}(S^2) \longrightarrow 0 \tag{5}$$

where $\Lambda_o^k(S^2)$ represents the closed k-forms on S^2 with integral periods. On S^2 all 3-forms are trivial. Therefore, for $k=2$ the exact sequence (5) implies the isomorphism

$$H^2(S^2, \mathbb{R}/\mathbb{Z}) \cong \hat{H}{}^2(S^2, \mathbb{R}/\mathbb{Z})$$

This is a generic characteristic the highest degree differential character groups, namely they reduce to the corresponding singular cohomology group of the same degree. The universal coefficient theorem gives

$$H^2(S^2, \mathbb{R}/\mathbb{Z}) = Hom(H_2(S^2, \mathbb{Z})) + Ext(H^1(S^2, \mathbb{Z}), U(1))$$

which reduces to

$$H^2(S^2, \mathbb{R}/\mathbb{Z}) = Hom(\mathbb{Z}, U(1))$$

One can also consider [5], [6] the exact sequence

$$0 \longrightarrow \Lambda^1(S^2)/\Lambda^1_o(S^2) \longrightarrow \overset{\wedge}{H^1} (S^2, \mathbb{R}/\mathbb{Z}) \longrightarrow H^2(S^2, \mathbb{Z}) \longrightarrow 0$$

where Λ^1, Λ^1_o are the spaces of closed one-forms and closed one-forms with integral periods respectively. Since all closed one-forms on S^2 have integral periods, the quotient space is trivial. This implies the isomorphism

$$\overset{\wedge}{H^1} (S^2, \mathbb{R}/\mathbb{Z}) \cong H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$$

which reproduces the conventional form of the Dirac quantization condition.

The same ideas can be reexpressed in a slightly different way as follows [6]: Consider the set

$$R^2(S^2, \mathbb{Z}) = \{(\omega, u) \in \Lambda^2_o \times H^2(M, \mathbb{Z}) | r(u) = [\omega]\}$$

with r being the natural map $r : H^2(S^2, \mathbb{Z}) \to H^2(S^2, \mathbb{R})$ and $[\omega]$ is the deRham class of ω. Generically that map has a large kernel. In this case the map is injective due to lack of torsion of $H^2(S^2, \mathbb{Z})$. Then the short exact sequence

$$0 \longrightarrow H^1(S^2, \mathbb{R})/r(H^1(S^2, \mathbb{Z})) \longrightarrow \overset{\wedge}{H^1} (S^2, \mathbb{R}/\mathbb{Z}) \longrightarrow R^2(S^2, \mathbb{Z}) \longrightarrow 0$$

gives rise to the isomorphism

$$\overset{\wedge}{H^1} (S^2, \mathbb{R}/\mathbb{Z}) \cong R^2(S^2, \mathbb{Z}) \tag{6}$$

Equation (6) also shows that the differential characters are generalizations of the differential forms, since the latter would have to vanish on the boundary of a simplex.

The geometric interpretation of these facts is the following: It is known [8] that there is an isomorphism

$$\hat{H}^1 (S^2, \mathbb{R}/\mathbb{Z}) \cong H^2(S^2, \mathbb{Z}(2)^\infty_D)$$

between the Cheeger-Simons differential 1-character group and the second smooth Deligne cohomology group with coefficients in the complex of sheaves $\mathbb{Z}(2)_D$. This complex of sheaves is defined to be the complex

$$\underline{\mathbb{Z}(2)}_{S^2} \longrightarrow \Lambda^0(S^2) \xrightarrow{d} \Lambda^1(S^2)$$

where $\mathbb{Z}(2) \cong (2\pi i)^2\mathbb{Z} \subset \mathbb{R}$ and $\underline{\mathbb{Z}(2)}$ is the constant sheaf corresponding to $\mathbb{Z}(2)$ [9]. The Deligne cohomology group $H^2(S^2, \mathbb{Z}(2)^\infty_D)$ is the set of isomorphism classes of smooth principal $U(1)$- bundles with connections over S^2 [10]. The curvatures of these connections are elements of $\Lambda^2_o(S^2)$ since the Bianchi identity $DF = 0$ reduces to $dF = 0$ for a connection on a $U(1)$-bundle on S^2, or more simply due to dimensional reasons. Furthermore, $H^1(S^2, U(1))$ is the set of isomorphism classes of flat connections on smooth $U(1)$-bundles over S^2. The exact sequence (5), for $k = 1$ expresses the fact that any isomorphism class of $U(1)$-bundles with a connection over S^2 can be obtained as the "twist" of a non-trivial $U(1)$-bundle with a connection, by a flat $U(1)$-connection.

We conclude that the differential characters provide a natural framework in which to express the Dirac quantization condition and they elucidate some associated geometric structures. Relatively recently, a theory of relative Cheeger-Simons differential

characters has been developed [11]. It would be interesting to see how this formalism can be used to describe other topological defects [12].

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